Anomalous dimensions of gauge fields and gauge coupling beta-functions in the Standard Model at three loops

A. V. Bednyakov[a], A. F. Pikelner[a] and V. N. Velizhanin[b]

[a] *Joint Institute for Nuclear Research*
141980 Dubna, Russia

[b] *Theoretical Physics Department, Petersburg Nuclear Physics Institute*
Orlova Roscha, Gatchina, 188300 St. Petersburg, Russia

Abstract

We present the results for three-loop gauge field anomalous dimensions in the SM calculated in the background field gauge within the unbroken phase of the model. The results are valid for the general background field gauge parameterized by three independent parameters. Both quantum and background fields are considered. The former are used to find three-loop anomalous dimensions for the gauge-fixing parameters, and the latter allow one to obtain the three-loop SM gauge beta-functions. Independence of beta-functions of gauge-fixing parameters serves as a validity check of our final results.

1 Introduction

In spite of the fact that the Standard Model has many unsatisfactory aspects Nature still does not allow us to find some solid evidence for the existence of a more fundamental theory with new particles and/or interactions. Due to the joint efforts of both experimentalists and theoreticians we are about to enter the only unexplored part of the SM and unveil the mechanism of electroweak symmetry breaking. According to the recent experimental results, there is strong evidence for the existence of the Higgs boson, the last missing ingredient of the SM spectrum [1, 2].

The mass of the higgs seems to be located at the boundary of the so-called stability and instability regions in the SM phase diagram. This fact implies that the SM can be potentially valid up to a very high scale (e.g., Plank scale).

In this situation, it is important to know how the running SM parameters evolve with energy scale. The analysis of high energy behavior is usually divided into two parts. The first one is the determination of running $\overline{\text{MS}}$-parameters from some (pseudo)observables. This procedure is usually referred to as "matching". The second one utilizes renormalization group equations (RGEs) to find the corresponding values at some "New Physics" scale. In order to carry out such an analysis consistently one usually use $(L-1)$-loop matching to find boundary conditions for L-loop RGEs (see, e.g., [3]). It is worth pointing that the advantage of the minimal-subtraction prescription lies in the fact that one needs to know only the ultraviolet (UV) divergent part of all the required diagrams. The latter has a simple polynomial structure in mass and momenta (once subdivergences are subtracted). Due to this, $\overline{\text{MS}}$ beta functions and anomalous dimensions can be relatively easily extracted from Green functions by solving a single scale problem with the help of the so-called infrared rearrangements (IRR) [4].

One- and two-loop results for SM beta functions have been known for quite a long time [5, 6, 7, 8, 9, 10, 11, 12, 13, 14] and are summarized in [15]. Until recently, three-loop corrections were known only partially [16, 17, 18, 19, 20, 21].

Having a well tested method for calculation of three-loop renormalization constants [22, 23, 24] and an experience in the calculations in the Standard Model and its minimal supersymmetric extension [25, 26, 27] we are planning to perform the calculation of all renormalization group coefficients in the third order of perturbation theory extending the results of Refs. [13, 28, 29]

to one more loop.

In this paper, we present our first step in this direction: the results for three-loop anomalous dimensions of the SM gauge fields. Since we are only interested in UV-divergences for the fields and dimensionless parameters, we do not consider the effects related to spontaneous breaking of electroweak symmetry and, as a consequence, can neglect all dimensionful parameters of the model. Moreover, we made use of the background-field gauge (BFG) (see, e.g., Ref. [30] and reference therein) to carry out our calculation. In this gauge, due to the simple QED-like Ward identities involving background fields, one can easily obtain expressions for the beta-functions by considering the two-point functions with external background particles.

During the work on this project a few papers on the same topic appeared [31, 32] (gauge couplings) and [33] (Top Yukawa and higgs self-interactions). Since the authors of [32] carried out a similar calculation, let us mention that our setup differs from that used in Ref. [31] in several aspects.

Firstly, for the diagram generation we made use of `FeynArts` [34], as it includes well-tested model files for the SM. Since the diagrams are evaluated with the help of the `MINCER` package [35], a mapping to the `MINCER` notation for momenta is required. This problem was solved by hand with the help of the `DIANA` [36] topology files which were prepared during our previous calculations [22]. Based on these files a simple script was written which allows one to perform the mapping between the `FeynArts` and `MINCER` notation.

Secondly, we not only consider Landau BFG in the broken phase of the SM[1], but also choose to work within the unbroken SM in a *general* BFG. The full dependence of the diagrams on the electroweak gauge-fixing parameters is retained and the corresponding renormalization is taken into account. And lastly, since the unbroken SM in BFG is not implemented as a `FeynArts` model file, we are forced to use a package like `FeynRules` [37] or `LanHEP` [38]. Due to the fact that the authors are more accustomed to the latter, it was chosen to generate the required Feynman rules from the Lagrangian.

The paper is organized as follows. In Section 2 we introduce our notation and present a brief description of the unbroken SM quantized in the background-field gauge. Section 3 describes the details of our calculation strategy. Finally, the results and conclusions can be found in Section 4. Appendix contains all the expressions for the considered renormalization constants.

[1] As a cross-check of our main calculation within the unbroken SM.

2 The Standard Model in the unbroken phase. The background-field gauge.

Let us briefly review the Lagrangian of the SM in the background-field gauge. We closely follow [39] albeit the fact that we introduce background fields only for gauge bosons. Moreover, as it was mentioned in Introduction, we neglect all the dimensionful couplings (i.e., mass parameters).

In our calculation we use the Lagrangian of the form

$$\mathcal{L} = \mathcal{L}_{\rm G} + \mathcal{L}_{\rm H} + \mathcal{L}_{\rm F} + \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP}. \tag{1}$$

Here $\mathcal{L}_{\rm G}$ is the Yang-Mills part

$$\begin{aligned}
\mathcal{L}_{\rm G} &= -\frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} - \frac{1}{4} W^i_{\mu\nu} W^i_{\mu\nu} - \frac{1}{4} B_{\mu\nu} B_{\mu\nu}, &(2)\\
G^a_{\mu\nu} &= \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu, &(3)\\
W^i_{\mu\nu} &= \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g_2 \epsilon^{ijk} W^j_\mu W^k_\nu, &(4)\\
B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, &(5)
\end{aligned}$$

where $G^a_\mu = \tilde{G}^a_\mu + \hat{G}^a_\mu$ $(a = 1, \ldots, 8)$, $W^i_\mu = \tilde{W}^i_\mu + \hat{W}^i_\mu$, $(i = 1, 2, 3)$, and $B_\mu = \tilde{B}_\mu + \hat{B}_\mu$ are gauge fields for SU(3), SU(2) and U(1) groups. By $\tilde{V} = (\tilde{G}, \tilde{W}, \tilde{B})$ we denote quantum fields, and $\hat{V} = (\hat{G}, \hat{W}, \hat{B})$ is used for their background counterpart. The corresponding gauge couplings are g_s, g_2, and g_1. The group structure constants enter into the commutation relations

$$\left[T^a, T^b\right] = i f^{abc} T^c, \qquad \left[\tau^i, \tau^j\right] = i\epsilon^{ijk} \tau^k, \tag{6}$$

with $T^a = \lambda^a/2$ and $\tau^i = \sigma^i/2$ being color and weak isospin generators.

The covariant derivative acting on a field which is charged under all the gauge groups looks like

$$D_\mu = \partial_\mu - g_s T^a G^a_\mu - g_2 \tau^i W^i_\mu + g_1 \frac{Y_W}{2} B_\mu. \tag{7}$$

If a field is not charged under either group, the corresponding term is omitted. With the help of the covariant derivative one can write the following Higgs

and fermionic parts of the Lagrangian:

$$\mathcal{L}_{\rm H} = (D_\mu \Phi)^\dagger (D_\mu \Phi) - \frac{\lambda}{2}\left(\Phi^\dagger \Phi\right)^2, \tag{8}$$

$$\begin{aligned}\mathcal{L}_{\rm F} = & \sum_{i=1,2,3}\left(i\bar{Q}^L_i \hat{D} Q^L_i + i\bar{L}^L_i \hat{D} L^L_i + i\bar{u}^R_g \hat{D} u^R_g + i\bar{d}^R_g \hat{D} d^R_g + i\bar{l}^R_g \hat{D} l^R_g\right) \\ & - \sum_{i,j=1,2,3}\left(Y^{ij}_u (Q^L_i \Phi^c) u^R_j + Y^{ij}_d (Q^L_i \Phi) d^R_j + Y^{ij}_l (L^L_i \Phi) l^R_j + \text{h.c.}\right),\end{aligned} \tag{9}$$

where indices $i, j = 1, 2, 3$ count different fermion families, λ and $Y_{u,d,l}$ are the higgs quartic and Yukawa matrices[2], respectively. The left-handed quarks $Q^L_g = (u_g, d_g)^L$ and leptons $L^L_g = (\nu_g, l_g)^L$ form the SU(2) doublets while the right-handed quarks (u^R_g, d^R_g) and charged leptons l^R_g are the singlets with respect to SU(2). The Higgs doublet Φ with $Y_W = 1$ has the following decomposition in terms of the component fields:

$$\Phi = \begin{pmatrix} \phi^+(x) \\ \frac{1}{\sqrt{2}}(h + i\chi) \end{pmatrix}, \qquad \Phi^c = i\sigma^2 \Phi^\dagger = \begin{pmatrix} \frac{1}{\sqrt{2}}(h - i\chi) \\ -\phi^- \end{pmatrix}. \tag{10}$$

Here a charge-conjugated Higgs doublet is introduced Φ^c with $Y_W = -1$.

The gauge-fixing terms are introduced only for quantum fields

$$\mathcal{L}_{\rm GF} = -\frac{1}{2\xi_G} G^a_G G^a_G - \frac{1}{2\xi_W} G^i_W G^i_W - \frac{1}{2\xi_B} G^2_B, \tag{11}$$

with

$$\begin{aligned} G^a_G &= \partial_\mu \tilde{G}^a_\mu + g_s f^{abc} \hat{G}^b_\mu \tilde{G}^c_\mu\,, \\ G^i_W &= \partial_\mu \tilde{W}^i_\mu + g_2 \epsilon^{ijk} \hat{W}^j_\mu \tilde{W}^k_\mu\,, \\ G_B &= \partial_\mu \tilde{B}_\mu\,. \end{aligned} \tag{12}$$

The ordinary derivatives are replaced by covariant ones containing the background fields. Due to this, the invariance of the effective action under background gauge transformations is not touched by introduction of (11).

The Fadeev-Popov part of the Lagrangian is given by

$$\mathcal{L}_{\rm FP} = -\bar{c}_\alpha \frac{\delta G_\alpha}{\delta \theta^\beta} c_\beta \tag{13}$$

[2]In the actual calculation the diagonal Yukawa matrices were used. However, the result can be generalized with the help of additional tricks (see Sec.3 and Ref. [32]).

where $\alpha, \beta = (G, W, B)$, and $\delta G_\alpha / \delta\theta^\beta$ is the variation of gauge-fixing functions (12) under the following infinitesimal quantum gauge transformations

$$
\begin{aligned}
\delta\tilde{G}^a_\mu &= (D_\mu \theta_G)^a = \partial_\mu \theta^a_G + g_s f^{abc} G^a_\mu \theta^a_G \,, \\
\delta\tilde{W}^i_\mu &= (D_\mu \theta_W)^i = \partial_\mu \theta^i_W + g_2 \epsilon^{ijk} W^i_\mu \theta^i_W \,, \\
\delta\tilde{B}_\mu &= \partial_\mu \theta_B \,.
\end{aligned} \tag{14}
$$

It should be stressed that covariant derivatives in (14) involve the sum of quantum and background gauge fields $V = \tilde{V} + \hat{V}$. The corresponding background transformations are obtained from (14) by the replacement $V \to \hat{V}$.

The Feynman rules for the model described by the Lagrangian (1) were generated with the help of `LanHEP` [3] [38].

It is worth mentioning here that our problem does not require the introduction of U(1) ghosts $\bar{c}_B, c_B$ and background $\hat{B}$ fields. This is due to the fact that the latter has the same interactions as its quantum counterpart $\tilde{B}$ and the former decouples from other particles. Nevertheless, we keep them in our `LanHEP` model file to allow for possible generalizations to non-linear gauge-fixing as in Ref. [39].

3 Details of calculations

Due to the gauge invariance of the effective action for the background fields, QED-like Ward identities can be derived. The latter can be used to prove the following simple relations:

$$
Z_{g_i} = Z_{\hat{V}_i}^{-1/2}, \qquad i = 1, 2, 3 \tag{15}
$$

with $Z_{\hat{V}_i}$ and Z_{g_i} being renormalization constants for background fields $\hat{V}^\mu_i = (\hat{B}^\mu, \hat{W}^\mu, \hat{G}^\mu)$ and SM gauge couplings $g_i = (g_1, g_2, g_s)$, respectively.

Since we keep the full dependence on the gauge-fixing parameters ξ_i during the whole calculation, we also need to know how $\xi_i = (\xi_B, \xi_W, \xi_G)$ are renormalized. Again, due to the Ward identities, the longitudinal part of the quantum gauge field propagators does not receive any loop corrections. As a consequence, the following identities hold:

$$
Z_{\xi_i} = Z_{\tilde{V}_i}. \tag{16}
$$

[3] `LanHEP` 3.1.5, which was used by the authors, produces a wrong sign for the combination $f^{abc} f^{dec}$ during export to the `FeynArts` model files. A new version with a fix is scheduled for November 2012.

Here Z_{ξ_i} stands for the renormalization constants for the gauge-fixing parameters. The quantum gauge fields $\tilde{V}_i$ are renormalized in the $\overline{\rm MS}$-scheme with the help of $Z_{\tilde{V}_i}$. It is clear from (15) and (16) that to carry out the calculation, one needs to consider gauge boson self-energies for both quantum $\tilde{V}$ and background $\hat{V}$ fields.

For calculation of the renormalization constants, following [19] (see also [8, 4, 40]), we use the multiplicative renormalizability of the corresponding Green functions. The renormalization constants Z_V relate the dimensionally regularized one-particle-irreducible two-point functions $\Gamma_{V,\rm Bare}$ with the renormalized one $\Gamma_{V,\rm Ren}$ as:

$$\Gamma_{V,\rm Ren}\left(\frac{Q^2}{\mu^2}, a_i\right) = \lim_{\epsilon\to 0} Z_V\left(\frac{1}{\epsilon}, a_i\right)\Gamma_{V,\rm Bare}\left(Q^2, a_{i,\rm Bare}, \epsilon\right), \qquad (17)$$

where $a_{i,\rm Bare}$ are the bare parameters of the model. For convenience, we introduce the following notation, which is closely related to that used in Ref. [32],

$$a_i = \left(\frac{5}{3}\frac{g_1^2}{16\pi^2}, \frac{g_2^2}{16\pi^2}, \frac{g_s^2}{16\pi^2}, \frac{Y_u^2}{16\pi^2}, \frac{Y_d^2}{16\pi^2}, \frac{Y_l^2}{16\pi^2}, \frac{\lambda}{16\pi^2}, \xi_G, \xi_W, \xi_G\right), \qquad (18)$$

so we treat the gauge-fixing parameters along the same lines as couplings. Moreover, in the renormalization group analysis of the SM one usually employs the SU(5) normalization of the U(1) gauge coupling which leads to an additional factor 5/3 in (18).

The bare parameters are related to the renormalized ones in the $\overline{\rm MS}$-scheme by the following formula:

$$a_{k,\rm Bare}\mu^{-2\rho_k\epsilon} = Z_{a_k} a_k(\mu) = a_k + \sum_{n=1}^{\infty} c_k^{(n)}\frac{1}{\epsilon^n}, \qquad (19)$$

where $\rho_k = 1$ for the gauge and Yukawa constants, $\rho_k = 2$ for the scalar quartic coupling constant, and $\rho_k = 0$ for the gauge fixing parameters. In order to extract a three-loop contribution to Z_V from the corresponding self-energies, it is sufficient to know the two-loop renormalization constants for the gauge couplings and the one-loop results for the Yukawa couplings. This is due to the fact that the Yukawa vertices appear for the first time only in the two-loop self-energies and the higgs self-coupling enters into the result only at the third level of perturbation theory.

The four-dimensional beta-functions, denoted by β_i, are defined via

$$\beta_i(a_k) = \left.\frac{da_i(\mu,\epsilon)}{d\ln\mu^2}\right|_{\epsilon=0} . \tag{20}$$

Here, again, a_i stands for both the gauge couplings and the gauge-fixing.

Given the fact that the bare parameters do not depend on the renormalization scale the expressions for β_i can be obtained [13] by differentiation of (19) with respect to $\ln\mu^2$:

$$-\rho_k\epsilon\left[a_k + \sum_{n=1}^{\infty} c_k^{(n)}\frac{1}{\epsilon^n}\right] = -\rho_k\epsilon a_k + \beta_k + \sum_{n=1}^{\infty}\sum_l(\beta_l - \rho_l a_l\epsilon)\frac{\partial c_k^{(n)}}{\partial a_l}\frac{1}{\epsilon^n} . \tag{21}$$

Taking in account only the leading order of the expansion in ϵ:

$$\beta_k = \sum_l \rho_l a_l \frac{\partial c_k^{(1)}}{\partial a_l} - \rho_k c_k^{(1)} . \tag{22}$$

In $\overline{\text{MS}}$-like schemes the renormalization constants for the Green functions may be expanded as

$$Z_\Gamma = 1 + \sum_{k=1}^{\infty}\frac{Z_\Gamma^{(k)}}{\epsilon^k} . \tag{23}$$

Differentiating (23) with respect to $\ln\mu^2$ we simply get all-order expression for anomalous dimensions:

$$\gamma_\Gamma \equiv -\mu^2\frac{\partial\ln Z_\Gamma}{\partial\mu^2} = -\left[\sum_j\left(\beta_j - \rho_j a_j\epsilon\right)\frac{\partial Z_\Gamma}{\partial a_j}\right] Z_\Gamma^{-1} . \tag{24}$$

It turns out that the above expression is finite as $\epsilon \to 0$ so

$$\gamma_\Gamma = \sum_j a_j\rho_j\frac{\partial Z_\Gamma^{(1)}}{\partial a_j} . \tag{25}$$

The advantage of (21) and (24) comes from the fact that it provides us with additional confirmation of the correctness of the final result since beta functions and anomalous dimensions extracted directly from (21) and (24) are finite for $\epsilon \to 0$ only if $c_k^{(n)}$ satisfy the so-called pole equations [41], e.g.,

$$\left[\sum_l \rho_l a_l\frac{\partial}{\partial a_l} - \rho_k\right] c_k^{(n+1)} = \sum_l \beta_l\frac{\partial c_k^{(n)}}{\partial a_l} . \tag{26}$$

In order to calculate bare the two-point functions for the quantum and background fields, we generate the corresponding diagrams with the help of the `FeynArts` package [34]. It is worth pointing that we use the `Classes` level of diagram generation which allows us to significantly reduce the number of generated diagrams since we do not distinguish fermion generations. The complexity of the problem can be deduced from Table 1 that shows how the number of the `FeynArts` generated diagrams increases with the loop level. Clearly, the presented numbers are an order of magnitude less than those given in Table I of Ref. [32], which somehow demonstrate the advantage of our approach.

The number of the SM fermion generations is introduced by hand via counting fermion traces present in the generated expression for a diagram and multiplying it by n_G. We separately count fermion traces involving the Yukawa interaction vertices and multiply them not by n_G but by n_Y. This allows us to use the following substitution rules (c.f., [32]) to generalize the obtained expression to the case of the general Yukawa matrices

$$\begin{aligned} n_Y\left[a_u, a_d, a_l\right] &\to \left[\mathcal{Y}_u, \mathcal{Y}_d, \mathcal{Y}_l\right], \\ n_Y\left[a_u^2, a_d^2, a_l^2\right] &\to \left[\mathcal{Y}_{uu}, \mathcal{Y}_{dd}, \mathcal{Y}_{ll}\right], \\ n_Y^2\left[a_u^2, a_d^2, a_l^2\right] &\to \left[\mathcal{Y}_u^2, \mathcal{Y}_d^2, \mathcal{Y}_l^2\right], \\ n_Y^2\left[a_u a_d, a_d a_l, a_u a_l\right] &\to \left[\mathcal{Y}_u\mathcal{Y}_d, \mathcal{Y}_d\mathcal{Y}_l, \mathcal{Y}_u\mathcal{Y}_l\right], \\ n_Y a_u a_d &\to \mathcal{Y}_{ud} \end{aligned} \tag{27}$$

where

$$\mathcal{Y}_u = \frac{\mathrm{tr}\, Y_u Y_u^\dagger}{16\pi^2}, \qquad \mathcal{Y}_d = \frac{\mathrm{tr}\, Y_d Y_d^\dagger}{16\pi^2}, \qquad \mathcal{Y}_l = \frac{\mathrm{tr}\, Y_l Y_l^\dagger}{16\pi^2}, \tag{28}$$

and

$$\begin{aligned} \mathcal{Y}_{uu} &= \frac{\mathrm{tr}\, Y_u Y_u^\dagger Y_u Y_u^\dagger}{(16\pi^2)^2}, \qquad & \mathcal{Y}_{dd} &= \frac{\mathrm{tr}\, Y_d Y_d^\dagger Y_d Y_d^\dagger}{(16\pi^2)^2}, \\ \mathcal{Y}_{ud} &= \frac{\mathrm{tr}\, Y_u Y_u^\dagger Y_d Y_d^\dagger}{(16\pi^2)^2}, \qquad & \mathcal{Y}_{ll} &= \frac{\mathrm{tr}\, Y_l Y_l^\dagger Y_l Y_l^\dagger}{(16\pi^2)^2}. \end{aligned} \tag{29}$$

A comment is in order about the last substitution in (27). It turns out that $\mathcal{Y}_{ud}$ is the only combination of up- and down-type Yukawa matrices, which can appear in the result for the three-loop gauge-boson self-energy within the SM. This can be traced to the following facts: 1) in the unbroken SM all the particles are massless so that chirality is conserved; 2) the

Broken	1	2	3	Unbroken	1	2	3
W^+/W^-	10	339	21942	$\hat{W}^i$	11	389	36647
Z	9	281	19041	$\tilde{W}^i$	11	371	36103
A	7	218	14426	$\hat{B}, \tilde{B}$	6	214	20144
ZA	7	236	16120	$\hat{G}$	4	73	4183
G	4	67	3287	$\tilde{G}$	4	66	4060
Total	37	1141	74816	Total	36	1113	101137

Table 1: Number of self-energy diagrams with external gauge fields, generated by `FeynArts` in the broken and unbroken SM, at one, two, and tree loops.

Yukawa interactions flip the chirality of the incoming fermions; 3) there is no right-handed flavour changing current coupled to a SM gauge field. As a consequence, combinations like

$$\frac{\mathrm{tr}\, Y_u Y_d^\dagger Y_u Y_d^\dagger}{(16\pi^2)^2} \qquad \text{and} \qquad \frac{\mathrm{tr}\, Y_u Y_d^\dagger Y_d Y_u^\dagger}{(16\pi^2)^2}, \tag{30}$$

which require at least two chirality-conserving transitions between right-handed up- and down-type quarks, do not show up in the result.

This type of counting is performed at the generation stage. A simple script converts the output of `FeynArts` to `DIANA`-like [36] notation and identifies `MINCER` topologies. This allows us to use the `FORM` [42] package `COLOR` [43] to do the color algebra and `MINCER` [35] to obtain the ϵ-expansion of diagrams. It is worth pointing that the expressions for all SM gauge couplings exhibit explicit dependence on number of colors N_c which stems from the fact that we have to sum over color when there is a (sub)loop with external color singlets coupled to quarks.

Having in mind the cancelation of gauge anomalies within the SM we use a naive anticommuting prescription for dealing with γ_5, so that all the Dirac traces involving one γ_5 vanish.

4 Results and conclusions

Here we present the results of our calculations in the form of the SM gauge beta-functions and anomalous dimension of the gauge-fixing parameters.

From (15) and (16) it is clear that anomalous dimensions of the background fields are connected with the corresponding gauge coupling beta-functions

$$\gamma_{\hat{B}} = -\beta_1/a_1, \qquad \gamma_{\hat{W}} = -\beta_2/a_2, \qquad \gamma_{\hat{G}} = -\beta_s/a_s \tag{31}$$

and for the quantum fields we have

$$\gamma_{\tilde{B}} = \beta_{\xi_B}/\xi_B, \qquad \gamma_{\tilde{W}} = \beta_{\xi_W}/\xi_W, \qquad \gamma_{\tilde{G}} = \beta_{\xi_G}/\xi_G. \tag{32}$$

The corresponding renormalization constants can be found in the Appendix.

At the end of the day, we have the following expressions for the beta-functions:

$$\begin{aligned}
\beta_1 = {} & a_1^2\left(n_G\left(\frac{11\ N_c}{45}+\frac{3}{5}\right)+\frac{1}{10}\right)\\
&+a_1^2\left(n_G\left(\frac{137a_1\ N_c}{900}+\frac{81a_1}{100}+\frac{a_2\ N_c}{20}+\frac{9a_2}{20}+\frac{11a_sC_F\ N_c}{15}\right)\right.\\
&\left.+\frac{9a_1}{50}+\frac{9\ a_2}{10}-\frac{N_c\mathcal{Y}_d}{6}-\frac{17N_c\ \mathcal{Y}_u}{30}-\frac{3\mathcal{Y}_l}{2}\right)\\
&+a_1^2\left(n_G\left(-\frac{1697a_1^2\ N_c}{18000}-\frac{981a_1^2}{2000}-\frac{a_1a_2\ N_c}{1200}-\frac{27a_1a_2}{400}\right.\right.\\
&-\frac{137}{900}\ a_1a_sC_FN_c+\frac{a_2^2\ N_c}{45}+\frac{27a_2^2}{10}-\frac{1}{20}a_2a_s\ C_FN_c+\frac{1463}{540}a_s^2C_AC_F\ N_c\\
&\left.-\frac{11}{30}a_s^2C_F^2N_c\right)+n_G^2\ \left(-\frac{16577a_1^2N_c^2}{486000}-\frac{2387a_1^2\ N_c}{9000}-\frac{891a_1^2}{2000}-\frac{11a_2^2\ N_c^2}{720}\right.\\
&\left.-\frac{11a_2^2N_c}{72}-\frac{11\ a_2^2}{80}-\frac{242}{135}a_s^2C_FT_F\ N_c\right)+\frac{489a_1^2}{8000}+\frac{783a_1\ a_2}{800}+\frac{27a_1\lambda}{50}\\
&-\frac{1267a_1\ N_c\mathcal{Y}_d}{2400}-\frac{2827a_1N_c\ \mathcal{Y}_u}{2400}-\frac{2529a_1\mathcal{Y}_l}{800}+\frac{3401\ a_2^2}{320}+\frac{9a_2\lambda}{10}\\
&-\frac{437a_2\ N_c\mathcal{Y}_d}{160}-\frac{157a_2N_c\ \mathcal{Y}_u}{32}-\frac{1629a_2\mathcal{Y}_l}{160}-\frac{17}{20}\ a_sC_FN_c\mathcal{Y}_d-\frac{29}{20}a_sC_F\ N_c\mathcal{Y}_u\\
&-\frac{9\lambda^2}{5}+\frac{17N_c^2\ \mathcal{Y}_d^2}{120}+\frac{59}{60}N_c^2\mathcal{Y}_d\mathcal{Y}_u+\frac{101\ N_c^2\mathcal{Y}_u^2}{120}+\frac{157N_c\mathcal{Y}_d\ \mathcal{Y}_l}{60}+\frac{61N_c\mathcal{Y}_{dd}}{80}\\
&\left.+\frac{199N_c\ \mathcal{Y}_l\mathcal{Y}_u}{60}+\frac{N_c\mathcal{Y}_{ud}}{8}+\frac{113\ N_c\mathcal{Y}_{uu}}{80}+\frac{99\mathcal{Y}_l^2}{40}+\frac{261\ \mathcal{Y}_{ll}}{80}\right),
\end{aligned}\tag{33}$$

$$
\begin{aligned}
\beta_2 = {} & a_2^2\biggl(n_G\,\left(\frac{N_c}{3}+\frac{1}{3}\right)-\frac{43}{6}\biggr) \\
& + a_2^2\biggl(n_G\biggl(\frac{a_1 N_c}{60}+\frac{3\ a_1}{20}+\frac{49 a_2 N_c}{12}+\frac{49\ a_2}{12}+a_s C_F N_c\biggr) \\
& + \frac{3\ a_1}{10}-\frac{259 a_2}{6}-\frac{N_c\ \mathcal{Y}_d}{2}-\frac{N_c \mathcal{Y}_u}{2}-\frac{\mathcal{Y}_l}{2}\biggr) \\
& + a_2^2\biggl(n_G\biggl(-\frac{287 a_1^2\ N_c}{3600}-\frac{91 a_1^2}{400}+\frac{13 a_1 a_2\ N_c}{240}+\frac{39 a_1 a_2}{80}-\frac{1}{60}a_1\ a_s C_F N_c \\
& + \frac{1603 a_2^2\ N_c}{27}+\frac{1603 a_2^2}{27}+\frac{13}{4}a_2\ a_s C_F N_c+\frac{133}{36}a_s^2 C_A\ C_F N_c-\frac{1}{2}a_s^2 C_F^2\ N_c\biggr) \\
& + n_G^2\biggl(-\frac{121 a_1^2\ N_c^2}{32400}-\frac{77 a_1^2 N_c}{1800}-\frac{33\ a_1^2}{400}-\frac{415 a_2^2 N_c^2}{432}-\frac{415\ a_2^2 N_c}{216}-\frac{415 a_2^2}{432} \\
& - \frac{22}{9}\ a_s^2 C_F T_F N_c\biggr)+\frac{163\ a_1^2}{1600}+\frac{561 a_1 a_2}{160}+\frac{3 a_1\ \lambda}{10}-\frac{533 a_1 N_c \mathcal{Y}_d}{480}-\frac{593\ a_1 N_c \mathcal{Y}_u}{480} \\
& - \frac{51 a_1\ \mathcal{Y}_l}{32}-\frac{667111 a_2^2}{1728}+\frac{3 a_2\ \lambda}{2}-\frac{243 a_2 N_c \mathcal{Y}_d}{32}-\frac{243\ a_2 N_c \mathcal{Y}_u}{32}-\frac{243 a_2\ \mathcal{Y}_l}{32} \\
& - \frac{7}{4}a_s C_F N_c\ \mathcal{Y}_d-\frac{7}{4}a_s C_F N_c \mathcal{Y}_u-3\ \lambda^2+\frac{5 N_c^2 \mathcal{Y}_d^2}{8}+\frac{5}{4}N_c^2\ \mathcal{Y}_d \mathcal{Y}_u+\frac{5 N_c^2 \mathcal{Y}_u^2}{8} \\
& + \frac{5 N_c\ \mathcal{Y}_d \mathcal{Y}_l}{4}+\frac{19 N_c \mathcal{Y}_{dd}}{16}+\frac{5\ N_c \mathcal{Y}_l \mathcal{Y}_u}{4}+\frac{9 N_c\ \mathcal{Y}_{ud}}{8}+\frac{19 N_c \mathcal{Y}_{uu}}{16} \\
& + \frac{5 \mathcal{Y}_l^2}{8}+\frac{19 \mathcal{Y}_{ll}}{16}\biggr),
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
\beta_s = {} & a_s^2\biggl(\frac{8 T_F n_G}{3}-\frac{11\ C_A}{3}\biggr) \\
& + a_s^2\biggl(n_G\biggl(\frac{11 a_1 T_F}{15}+3\ a_2 T_F+\frac{40 a_s C_A T_F}{3}+8\ a_s C_F T_F\biggr) \\
& - \frac{34 a_s C_A^2}{3}-4\ T_F \mathcal{Y}_d-4 T_F \mathcal{Y}_u\biggr)+a_s^2\biggl(n_G\biggl(-\frac{13 a_1^2\ T_F}{60}-\frac{a_1 a_2 T_F}{20} \\
& + \frac{22}{15}\ a_1 a_s C_A T_F-\frac{11}{15}a_1\ a_s C_F T_F+\frac{241 a_2^2 T_F}{12} \\
& + 6 a_2 a_s C_A T_F-3 a_2 a_s C_F\ T_F+\frac{2830}{27}a_s^2 C_A^2 T_F+\frac{410}{9}\ a_s^2 C_A C_F T_F-4 a_s^2 C_F^2\ T_F\biggr)
\end{aligned}
$$

$$
\begin{aligned}
&+ n_G^2\bigg(- \frac{1331a_1^2T_F\ N_c}{8100} - \frac{121a_1^2T_F}{300} - \frac{11}{12}\ a_2^2T_FN_c - \frac{11a_2^2\ T_F}{12} - \frac{632}{27}a_s^2C_AT_F^2 \\
&- \frac{176}{9}a_s^2C_FT_F^2\bigg) - \frac{89a_1T_F\mathcal{Y}_d}{20} - \frac{101\ a_1T_F\mathcal{Y}_u}{20} - \frac{93a_2T_F\ \mathcal{Y}_d}{4} - \frac{93a_2T_F\mathcal{Y}_u}{4} \\
&- \frac{2857}{54}\ a_s^2C_A^3 - 24a_sC_AT_F\mathcal{Y}_d - 24\ a_sC_AT_F\mathcal{Y}_u - 6a_sC_FT_F\ \mathcal{Y}_d - 6a_sC_FT_F\mathcal{Y}_u \\
&+ 7T_F\ N_c\mathcal{Y}_d^2 + 14T_FN_c\mathcal{Y}_d\mathcal{Y}_u + 7\ T_FN_c\mathcal{Y}_u^2 + 7T_F\mathcal{Y}_d\mathcal{Y}_l + 9\ T_F\mathcal{Y}_{dd} + 7T_F\mathcal{Y}_l\mathcal{Y}_u \\
&- 6T_F\ \mathcal{Y}_{ud} + 9T_F\mathcal{Y}_{uu}\bigg)\,, \qquad (35)
\end{aligned}
$$

$$
\begin{aligned}
\beta_{\xi_B} = {}& \xi_B\bigg(- \frac{a_1}{10} + n_G\bigg(- \frac{3a_1}{5} - \frac{11a_1N_c}{45}\bigg)\bigg) \\
&+ \xi_B\bigg(- \frac{9a_1^2}{50} - \frac{9a_1a_2}{10} + \frac{a_1N_c\mathcal{Y}_d}{6} + \frac{3a_1\mathcal{Y}_l}{2} + \frac{17a_1N_c\mathcal{Y}_u}{30} \\
&+ n_G\bigg(- \frac{81a_1^2}{100} - \frac{9a_1a_2}{20} - \frac{137a_1^2N_c}{900} - \frac{a_1a_2N_c}{20} - \frac{11}{15}a_1a_sC_FN_c\bigg)\bigg) \\
&+ \xi_B\bigg(- \frac{489a_1^3}{8000} - \frac{783a_1^2a_2}{800} - \frac{3401a_1a_2^2}{320} - \frac{54a_1^2\lambda}{25} - \frac{18a_1a_2\lambda}{5} + \frac{144a_1\lambda^2}{5} \\
&+ n_G\bigg(\frac{981a_1^3}{2000} + \frac{27a_1^2a_2}{400} - \frac{27a_1a_2^2}{10} + \frac{1697a_1^3N_c}{18000} + \frac{a_1^2a_2N_c}{1200} - \frac{1}{45}a_1a_2^2N_c \\
&+ \frac{137}{900}a_1^2a_sC_FN_c + \frac{1}{20}a_1a_2a_sC_FN_c - \frac{1463}{540}a_1a_s^2C_AC_FN_c + \frac{11}{30}a_1a_s^2C_F^2N_c\bigg) \\
&+ n_G^2\bigg(\frac{891a_1^3}{2000} + \frac{11a_1a_2^2}{80} + \frac{2387a_1^3N_c}{9000} + \frac{11}{72}a_1a_2^2N_c + \frac{242}{135}a_1a_s^2C_FT_FN_c \\
&+ \frac{16577a_1^3N_c^2}{486000} + \frac{11}{720}a_1a_2^2N_c^2\bigg) + \frac{1267a_1^2N_c\mathcal{Y}_d}{2400} + \frac{437}{160}a_1a_2N_c\mathcal{Y}_d \\
&+ \frac{17}{20}a_1a_sC_FN_c\mathcal{Y}_d - \frac{17}{120}a_1N_c^2\mathcal{Y}_d^2 - \frac{61a_1N_c\mathcal{Y}_{dd}}{80} + \frac{2529a_1^2\mathcal{Y}_l}{800} + \frac{1629a_1a_2\mathcal{Y}_l}{160} \\
&- \frac{157}{60}a_1N_c\mathcal{Y}_d\mathcal{Y}_l - \frac{99a_1\mathcal{Y}_l^2}{40} - \frac{261a_1\mathcal{Y}_{ll}}{80} + \frac{2827a_1^2N_c\mathcal{Y}_u}{2400} + \frac{157}{32}a_1a_2N_c\mathcal{Y}_u \\
&+ \frac{29}{20}a_1a_sC_FN_c\mathcal{Y}_u - \frac{59}{60}a_1N_c^2\mathcal{Y}_d\mathcal{Y}_u - \frac{199}{60}a_1N_c\mathcal{Y}_l\mathcal{Y}_u - \frac{101}{120}a_1N_c^2\mathcal{Y}_u^2 \\
&- \frac{a_1N_c\mathcal{Y}_{ud}}{8} - \frac{113a_1N_c\mathcal{Y}_{uu}}{80}\bigg)\,, \qquad (36)
\end{aligned}
$$

$$
\begin{aligned}
\beta_{\xi_W} &= \xi_W\bigg(\frac{25a_2}{6} - a_2\xi_W + n_G\bigg(-\frac{a_2}{3} - \frac{a_2N_c}{3}\bigg)\bigg) \\
&+ \xi_W\bigg(-\frac{3a_1a_2}{10} + \frac{113a_2^2}{4} - \frac{11a_2^2\xi_W}{2} - a_2^2\xi_W^2 + n_G\bigg(-\frac{3a_1a_2}{20} \\
&- \frac{13a_2^2}{4} - \frac{a_1a_2N_c}{60} - \frac{13a_2^2N_c}{4} - a_2a_sC_FN_c\bigg) + \frac{a_2N_c\mathcal{Y}_d}{2} + \frac{a_2\mathcal{Y}_l}{2} + \frac{a_2N_c\mathcal{Y}_u}{2}\bigg) \\
&+ \xi_W\bigg(-\frac{163a_1^2a_2}{1600} - \frac{33a_1a_2^2}{32} + \frac{143537a_2^3}{576} - \frac{6a_1a_2\lambda}{5} - 6a_2^2\lambda + 48a_2\lambda^2 \\
&- \frac{315a_2^3\xi_W}{8} - \frac{33a_2^3\xi_W^2}{4} - \frac{7a_2^3\xi_W^3}{4} + n_G^2\bigg(\frac{33a_1^2a_2}{400} + \frac{185a_2^3}{144} + \frac{77a_1^2a_2N_c}{1800} \\
&+ \frac{185a_2^3N_c}{72} + \frac{22}{9}a_2a_s^2C_FT_FN_c + \frac{121a_1^2a_2N_c^2}{32400} + \frac{185a_2^3N_c^2}{144}\bigg) + \frac{533}{480}a_1a_2N_c\mathcal{Y}_d \\
&+ \frac{79}{32}a_2^2N_c\mathcal{Y}_d + \frac{7}{4}a_2a_sC_FN_c\mathcal{Y}_d - \frac{5}{8}a_2N_c^2\mathcal{Y}_d^2 - \frac{19a_2N_c\mathcal{Y}_{dd}}{16} + \frac{51a_1a_2\mathcal{Y}_l}{32} \\
&+ \frac{79a_2^2\mathcal{Y}_l}{32} - \frac{5}{4}a_2N_c\mathcal{Y}_d\mathcal{Y}_l - \frac{5a_2\mathcal{Y}_l^2}{8} - \frac{19a_2\mathcal{Y}_{ll}}{16} + \frac{593}{480}a_1a_2N_c\mathcal{Y}_u + \frac{79}{32}a_2^2N_c\mathcal{Y}_u \\
&+ \frac{7}{4}a_2a_sC_FN_c\mathcal{Y}_u - \frac{5}{4}a_2N_c^2\mathcal{Y}_d\mathcal{Y}_u - \frac{5}{4}a_2N_c\mathcal{Y}_l\mathcal{Y}_u - \frac{5}{8}a_2N_c^2\mathcal{Y}_u^2 - \frac{9a_2N_c\mathcal{Y}_{ud}}{8} \\
&- \frac{19a_2N_c\mathcal{Y}_{uu}}{16} - \frac{9}{10}a_1a_2^2\zeta(3) + \frac{3a_2^3\zeta(3)}{2} - 6a_2^3\xi_W\zeta(3) - \frac{3}{2}a_2^3\xi_W^2\zeta(3) \\
&+ n_G\bigg(\frac{91a_1^2a_2}{400} + \frac{6a_1a_2^2}{5} - \frac{7025a_2^3}{144} + 2a_2^3\xi_W + \frac{287a_1^2a_2N_c}{3600} + \frac{2}{15}a_1a_2^2N_c \\
&- \frac{7025a_2^3N_c}{144} + \frac{1}{60}a_1a_2a_sC_FN_c + 8a_2^2a_sC_FN_c - \frac{133}{36}a_2a_s^2C_AC_FN_c \\
&+ \frac{1}{2}a_2a_s^2C_F^2N_c + 2a_2^3\xi_WN_c - \frac{9}{5}a_1a_2^2\zeta(3) + 9a_2^3\zeta(3) - \frac{1}{5}a_1a_2^2N_c\zeta(3) \\
&+ 9a_2^3N_c\zeta(3) - 12a_2^2a_sC_FN_c\zeta(3)\bigg)\bigg)\,, \qquad (37)
\end{aligned}
$$

$$
\begin{aligned}
\beta_{\xi_G} &= \xi_G\bigg(\frac{13a_sC_A}{6} - \frac{a_sC_A\xi_G}{2} - \frac{8a_sT_Fn_G}{3}\bigg) \\
&+ \xi_G\bigg(\frac{59a_s^2C_A^2}{8} - \frac{11}{8}a_s^2C_A^2\xi_G - \frac{1}{4}a_s^2C_A^2\xi_G^2 + 4a_sT_F\mathcal{Y}_d + 4a_sT_F\mathcal{Y}_u
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(-\frac{11}{15}a_1a_sT_F-3a_2a_sT_F-10a_s^2C_AT_F-8a_s^2C_FT_F\Big)n_G\Big)\\
&+\xi_G\Big(\frac{9965a_s^3C_A^3}{288}-\frac{167}{32}a_s^3C_A^3\xi_G-\frac{33}{32}a_s^3C_A^3\xi_G^2-\frac{7}{32}a_s^3C_A^3\xi_G^3+n_G^2\Big(\frac{121}{300}a_1^2a_sT_F\\
&+\frac{11}{12}a_2^2a_sT_F+\frac{304}{9}a_s^3C_AT_F^2+\frac{176}{9}a_s^3C_FT_F^2+\frac{1331a_1^2a_sT_FN_c}{8100}+\frac{11}{12}a_2^2a_sT_FN_c\Big)\\
&+\frac{89}{20}a_1a_sT_F\mathcal{Y}_d+\frac{93}{4}a_2a_sT_F\mathcal{Y}_d+\frac{25}{2}a_s^2C_AT_F\mathcal{Y}_d+6a_s^2C_FT_F\mathcal{Y}_d-7a_sT_FN_c\mathcal{Y}_d^2\\
&-9a_sT_F\mathcal{Y}_{dd}-7a_sT_F\mathcal{Y}_d\mathcal{Y}_l+\frac{101}{20}a_1a_sT_F\mathcal{Y}_u+\frac{93}{4}a_2a_sT_F\mathcal{Y}_u+\frac{25}{2}a_s^2C_AT_F\mathcal{Y}_u\\
&+6a_s^2C_FT_F\mathcal{Y}_u-14a_sT_FN_c\mathcal{Y}_d\mathcal{Y}_u-7a_sT_F\mathcal{Y}_l\mathcal{Y}_u-7a_sT_FN_c\mathcal{Y}_u^2+6a_sT_F\mathcal{Y}_{ud}\\
&-9a_sT_F\mathcal{Y}_{uu}-\frac{9}{16}a_s^3C_A^3\zeta(3)-\frac{3}{4}a_s^3C_A^3\xi_G\zeta(3)-\frac{3}{16}a_s^3C_A^3\xi_G^2\zeta(3)+n_G\Big(\frac{13}{60}a_1^2a_sT_F\\
&+\frac{1}{20}a_1a_2a_sT_F-\frac{241}{12}a_2^2a_sT_F+\frac{319}{120}a_1a_s^2C_AT_F+\frac{87}{8}a_2a_s^2C_AT_F-\frac{911}{9}a_s^3C_A^2T_F\\
&+\frac{11}{15}a_1a_s^2C_FT_F+3a_2a_s^2C_FT_F-\frac{5}{9}a_s^3C_AC_FT_F+4a_s^3C_F^2T_F+4a_s^3C_A^2\xi_GT_F\\
&-\frac{22}{5}a_1a_s^2C_AT_F\zeta(3)-18a_2a_s^2C_AT_F\zeta(3)+36a_s^3C_A^2T_F\zeta(3)\\
&-48a_s^3C_AC_FT_F\zeta(3)\Big)\Big)\,. \qquad (38)
\end{aligned}
$$

With the help of substitutions $C_A=N_c=3$, $C_F=4/3$, $T_F=1/2$, $\mathcal{Y}_u=\operatorname{tr}\hat{T}$, $\mathcal{Y}_d=\operatorname{tr}\hat{B}$, $\mathcal{Y}_l=\operatorname{tr}\hat{L}$, $\mathcal{Y}_{dd}=\operatorname{tr}(\hat{B}^2)$, $\mathcal{Y}_{uu}=\operatorname{tr}(\hat{T}^2)$, $\mathcal{Y}_{ll}=\operatorname{tr}(\hat{T}^2)$, and $\mathcal{Y}_{ud}=\operatorname{tr}\hat{T}\hat{B}$ it is possible to prove that the expressions presented above coincide with the results for the gauge beta functions obtained in Ref. [31].

As a consequence, one can be sure that the three-loop renormalization group equations obtained for the first time in Ref. [31] are correct and confirmed by an independent calculation. It is also worth mentioning that the obtained results can be used not only for the analysis of vacuum stability constraints within the SM (see, e.g., [44, 45, 46]) but also, e.g., for very precise matching of the SM with its supersymmetric extension since the corresponding three-loop renormalization group functions are already known from the literature [47, 48]. Moreover, the leading two-loop decoupling corrections for the strongest SM couplings are also calculated within the MSSM in Refs. [25, 49, 50, 26].

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A Renormalization constants

Here we present the results for the renormalization constants from which the anomalous dimensions and beta-functions were extracted. It should be pointed out that the coefficients of the ϵ-expansion satisfy the pole equations (26). The corresponding expressions together with the results for beta-functions can be found online[4] in the form of `Mathematica` files.

$$
\begin{aligned}
Z_{\alpha_1} = 1 &+ a_1\frac{1}{\epsilon}\left\{n_G\left(\frac{11N_c}{45}+\frac{3}{5}\right)+\frac{1}{10}\right\} \\
&+ a_1\left\{\frac{1}{\epsilon^2}\left[n_G^2\left(\frac{121a_1N_c^2}{2025}+\frac{22a_1N_c}{75}+\frac{9a_1}{25}\right)+n_G\left(\frac{11a_1N_c}{225}+\frac{3a_1}{25}\right)+\frac{a_1}{100}\right]\right. \\
&+ \frac{1}{\epsilon}\left[n_G\left(\frac{137a_1N_c}{1800}+\frac{81a_1}{200}+\frac{a_2N_c}{40}+\frac{9a_2}{40}+\frac{11a_sC_FN_c}{30}\right)+\frac{9a_1}{100}+\frac{9a_2}{20}\right. \\
&\left.\left.- \frac{N_c\mathcal{Y}_d}{12}-\frac{17N_c\mathcal{Y}_u}{60}-\frac{3\mathcal{Y}_l}{4}\right]\right\} \\
&+ a_1\left\{\frac{1}{\epsilon^3}\left[n_G^3\left(\frac{1331a_1^2N_c^3}{91125}+\frac{121a_1^2N_c^2}{1125}+\frac{33a_1^2N_c}{125}+\frac{27a_1^2}{125}\right)\right.\right. \\
&\left.+ n_G^2\left(\frac{121a_1^2N_c^2}{6750}+\frac{11a_1^2N_c}{125}+\frac{27a_1^2}{250}\right)+n_G\left(\frac{11a_1^2N_c}{1500}+\frac{9a_1^2}{500}\right)+\frac{a_1^2}{1000}\right] \\
&+ \frac{1}{\epsilon^2}\left[n_G\left(\frac{3731a_1^2N_c}{54000}+\frac{441a_1^2}{2000}+\frac{9a_1a_2N_c}{40}+\frac{117a_1a_2}{200}+\frac{11}{150}a_1a_sC_FN_c\right.\right. \\
&- \frac{11}{270}a_1N_c^2\mathcal{Y}_d-\frac{187a_1N_c^2\mathcal{Y}_u}{1350}-\frac{a_1N_c\mathcal{Y}_d}{10}-\frac{11a_1N_c\mathcal{Y}_l}{30}-\frac{17a_1N_c\mathcal{Y}_u}{50}-\frac{9a_1\mathcal{Y}_l}{10} \\
&\left.- \frac{7a_2^2N_c}{720}-\frac{39a_2^2}{80}-\frac{121}{270}a_s^2C_AC_FN_c\right)+n_G^2\left(\frac{10549a_1^2N_c^2}{243000}+\frac{1519a_1^2N_c}{4500}\right.
\end{aligned}
$$

[4] As ancillary files of the `arXiv` version of the paper

$$
\begin{aligned}
&+\frac{567a_1^2}{1000}+\frac{11}{900}a_1a_2N_c^2+\frac{7a_1a_2N_c}{50}+\frac{27a_1a_2}{100}+\frac{121}{675}a_1a_sC_FN_c^2+\frac{11}{25}a_1a_sC_FN_c\\
&+\frac{a_2^2N_c^2}{360}+\frac{a_2^2N_c}{36}+\frac{a_2^2}{40}+\frac{44}{135}a_s^2C_FT_FN_c\Bigg)+\frac{21a_1^2}{1000}+\frac{9a_1a_2}{100}-\frac{7a_1N_c\mathcal{Y}_d}{720}\\
&+\frac{17a_1N_c\mathcal{Y}_u}{720}+\frac{33a_1\mathcal{Y}_l}{80}-\frac{43a_2^2}{40}+\frac{a_2N_c\mathcal{Y}_d}{16}+\frac{17a_2N_c\mathcal{Y}_u}{80}+\frac{9a_2\mathcal{Y}_l}{16}\\
&+\frac{1}{6}a_sC_FN_c\mathcal{Y}_d+\frac{17}{30}a_sC_FN_c\mathcal{Y}_u-\frac{N_c^2\mathcal{Y}_d^2}{36}-\frac{11}{90}N_c^2\mathcal{Y}_d\mathcal{Y}_u-\frac{17N_c^2\mathcal{Y}_u^2}{180}\\
&-\frac{5N_c\mathcal{Y}_d\mathcal{Y}_l}{18}-\frac{N_c\mathcal{Y}_{dd}}{24}-\frac{31N_c\mathcal{Y}_l\mathcal{Y}_u}{90}+\frac{11N_c\mathcal{Y}_{ud}}{60}-\frac{17N_c\mathcal{Y}_{uu}}{120}-\frac{\mathcal{Y}_l^2}{4}-\frac{3\mathcal{Y}_{ll}}{8}\Bigg]\\
&+\frac{1}{\epsilon}\Bigg[n_G\Bigg(-\frac{1697a_1^2N_c}{54000}-\frac{327a_1^2}{2000}-\frac{a_1a_2N_c}{3600}-\frac{9a_1a_2}{400}-\frac{137a_1a_sC_FN_c}{2700}\\
&+\frac{a_2^2N_c}{135}+\frac{9a_2^2}{10}-\frac{1}{60}a_2a_sC_FN_c+\frac{1463a_s^2C_AC_FN_c}{1620}-\frac{11}{90}a_s^2C_F^2N_c\Bigg)\\
&+n_G^2\Bigg(-\frac{16577a_1^2N_c^2}{1458000}-\frac{2387a_1^2N_c}{27000}-\frac{297a_1^2}{2000}-\frac{11a_2^2N_c^2}{2160}-\frac{11a_2^2N_c}{216}-\frac{11a_2^2}{240}\\
&-\frac{242}{405}a_s^2C_FT_FN_c\Bigg)+\frac{163a_1^2}{8000}+\frac{261a_1a_2}{800}+\frac{9a_1\lambda}{50}-\frac{1267a_1N_c\mathcal{Y}_d}{7200}\\
&-\frac{2827a_1N_c\mathcal{Y}_u}{7200}-\frac{843a_1\mathcal{Y}_l}{800}+\frac{3401a_2^2}{960}+\frac{3a_2\lambda}{10}-\frac{437a_2N_c\mathcal{Y}_d}{480}-\frac{157a_2N_c\mathcal{Y}_u}{96}\\
&-\frac{543a_2\mathcal{Y}_l}{160}-\frac{17}{60}a_sC_FN_c\mathcal{Y}_d-\frac{29}{60}a_sC_FN_c\mathcal{Y}_u-\frac{3\lambda^2}{5}+\frac{17N_c^2\mathcal{Y}_d^2}{360}+\frac{59}{180}N_c^2\mathcal{Y}_d\mathcal{Y}_u\\
&+\frac{101N_c^2\mathcal{Y}_u^2}{360}+\frac{157N_c\mathcal{Y}_d\mathcal{Y}_l}{180}+\frac{61N_c\mathcal{Y}_{dd}}{240}+\frac{199N_c\mathcal{Y}_l\mathcal{Y}_u}{180}+\frac{N_c\mathcal{Y}_{ud}}{24}+\frac{113N_c\mathcal{Y}_{uu}}{240}\\
&+\frac{33\mathcal{Y}_l^2}{40}+\frac{87\mathcal{Y}_{ll}}{80}\Bigg]\Bigg\}, \qquad (39)
\end{aligned}
$$

$$
\begin{aligned}
Z_{\alpha_2}&=1+a_2\frac{1}{\epsilon}\Bigg\{n_G\Bigg(\frac{N_c}{3}+\frac{1}{3}\Bigg)-\frac{43}{6}\Bigg\}\\
&+a_2\Bigg\{\frac{1}{\epsilon^2}\Bigg[n_G^2\Bigg(\frac{a_2N_c^2}{9}+\frac{2a_2N_c}{9}+\frac{a_2}{9}\Bigg)+n_G\Bigg(-\frac{43a_2N_c}{9}-\frac{43a_2}{9}\Bigg)+\frac{1849a_2}{36}\Bigg]\\
&+\frac{1}{\epsilon}\Bigg[n_G\Bigg(\frac{a_1N_c}{120}+\frac{3a_1}{40}+\frac{49a_2N_c}{24}+\frac{49a_2}{24}+\frac{a_sC_FN_c}{2}\Bigg)\\
&+\frac{3a_1}{20}-\frac{259a_2}{12}-\frac{N_c\mathcal{Y}_d}{4}-\frac{N_c\mathcal{Y}_u}{4}-\frac{\mathcal{Y}_l}{4}\Bigg]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+ a_2\Bigg\{\frac{1}{\epsilon^3}\Bigg[n_G\bigg(\frac{1849a_2^2N_c}{36}+\frac{1849a_2^2}{36}\bigg)+n_G^2\bigg(-\frac{43}{18}a_2^2N_c^2-\frac{43a_2^2N_c}{9}-\frac{43a_2^2}{18}\bigg)\\
&+n_G^3\bigg(\frac{a_2^2N_c^3}{27}+\frac{a_2^2N_c^2}{9}+\frac{a_2^2N_c}{9}+\frac{a_2^2}{27}\bigg)-\frac{79507a_2^2}{216}\Bigg]+\frac{1}{\epsilon^2}\Bigg[n_G\bigg(\frac{a_1^2N_c}{80}+\frac{13a_1^2}{400}\\
&-\frac{7a_1a_2N_c}{360}-\frac{39a_1a_2}{40}-\frac{22001a_2^2N_c}{432}-\frac{22001a_2^2}{432}-\frac{43}{6}a_2a_sC_FN_c-\frac{1}{6}a_2N_c^2\mathcal{Y}_d\\
&-\frac{1}{6}a_2N_c^2\mathcal{Y}_u-\frac{a_2N_c\mathcal{Y}_d}{6}-\frac{a_2N_c\mathcal{Y}_l}{6}-\frac{a_2N_c\mathcal{Y}_u}{6}-\frac{a_2\mathcal{Y}_l}{6}-\frac{11}{18}a_s^2C_AC_FN_c\bigg)\\
&+n_G^2\bigg(\frac{11a_1^2N_c^2}{16200}+\frac{7a_1^2N_c}{900}+\frac{3a_1^2}{200}+\frac{1}{180}a_1a_2N_c^2+\frac{a_1a_2N_c}{18}+\frac{a_1a_2}{20}+\frac{343a_2^2N_c^2}{216}\\
&+\frac{343a_2^2N_c}{108}+\frac{343a_2^2}{216}+\frac{1}{3}a_2a_sC_FN_c^2+\frac{1}{3}a_2a_sC_FN_c+\frac{4}{9}a_s^2C_FT_FN_c\bigg)+\frac{a_1^2}{200}\\
&-\frac{43a_1a_2}{20}+\frac{a_1N_c\mathcal{Y}_d}{48}+\frac{17a_1N_c\mathcal{Y}_u}{240}+\frac{3a_1\mathcal{Y}_l}{16}+\frac{77959a_2^2}{216}+\frac{181a_2N_c\mathcal{Y}_d}{48}\\
&+\frac{181a_2N_c\mathcal{Y}_u}{48}+\frac{181a_2\mathcal{Y}_l}{48}+\frac{1}{2}a_sC_FN_c\mathcal{Y}_d+\frac{1}{2}a_sC_FN_c\mathcal{Y}_u-\frac{N_c^2\mathcal{Y}_d^2}{12}-\frac{1}{6}N_c^2\mathcal{Y}_d\mathcal{Y}_u\\
&-\frac{N_c^2\mathcal{Y}_u^2}{12}-\frac{N_c\mathcal{Y}_d\mathcal{Y}_l}{6}-\frac{N_c\mathcal{Y}_{dd}}{8}-\frac{N_c\mathcal{Y}_l\mathcal{Y}_u}{6}+\frac{N_c\mathcal{Y}_{ud}}{4}-\frac{N_c\mathcal{Y}_{uu}}{8}-\frac{\mathcal{Y}_l^2}{12}-\frac{\mathcal{Y}_{ll}}{8}\Bigg]\\
&+\frac{1}{\epsilon}\Bigg[n_G\bigg(-\frac{287a_1^2N_c}{10800}-\frac{91a_1^2}{1200}+\frac{13a_1a_2N_c}{720}+\frac{13a_1a_2}{80}-\frac{1}{180}a_1a_sC_FN_c\\
&+\frac{1603a_2^2N_c}{81}+\frac{1603a_2^2}{81}+\frac{13}{12}a_2a_sC_FN_c+\frac{133}{108}a_s^2C_AC_FN_c-\frac{1}{6}a_s^2C_F^2N_c\bigg)\\
&+n_G^2\bigg(-\frac{121a_1^2N_c^2}{97200}-\frac{77a_1^2N_c}{5400}-\frac{11a_1^2}{400}-\frac{415a_2^2N_c^2}{1296}-\frac{415a_2^2N_c}{648}-\frac{415a_2^2}{1296}\\
&-\frac{22}{27}a_s^2C_FT_FN_c\bigg)+\frac{163a_1^2}{4800}+\frac{187a_1a_2}{160}+\frac{a_1\lambda}{10}-\frac{533a_1N_c\mathcal{Y}_d}{1440}-\frac{593a_1N_c\mathcal{Y}_u}{1440}\\
&-\frac{17a_1\mathcal{Y}_l}{32}-\frac{667111a_2^2}{5184}+\frac{a_2\lambda}{2}-\frac{81a_2N_c\mathcal{Y}_d}{32}-\frac{81a_2N_c\mathcal{Y}_u}{32}-\frac{81a_2\mathcal{Y}_l}{32}\\
&-\frac{7}{12}a_sC_FN_c\mathcal{Y}_d-\frac{7}{12}a_sC_FN_c\mathcal{Y}_u-\lambda^2+\frac{5N_c^2\mathcal{Y}_d^2}{24}+\frac{5}{12}N_c^2\mathcal{Y}_d\mathcal{Y}_u+\frac{5N_c^2\mathcal{Y}_u^2}{24}\\
&+\frac{5N_c\mathcal{Y}_d\mathcal{Y}_l}{12}+\frac{19N_c\mathcal{Y}_{dd}}{48}+\frac{5N_c\mathcal{Y}_l\mathcal{Y}_u}{12}+\frac{3N_c\mathcal{Y}_{ud}}{8}+\frac{19N_c\mathcal{Y}_{uu}}{48}\\
&+\frac{5\mathcal{Y}_l^2}{24}+\frac{19\mathcal{Y}_{ll}}{48}\Bigg]\Bigg\}, \qquad (40)
\end{aligned}
$$

$$
\begin{aligned}
Z_{\alpha_s} &= 1 + a_s\frac{1}{\epsilon}\left\{\frac{8T_Fn_G}{3} - \frac{11C_A}{3}\right\} \\
&+ a_s\left\{\frac{1}{\epsilon^2}\left[\frac{121a_sC_A^2}{9} - \frac{176}{9}a_sC_AT_Fn_G + \frac{64}{9}a_sT_F^2n_G^2\right]\right. \\
&+ \frac{1}{\epsilon}\left[n_G\left(\frac{11a_1T_F}{30} + \frac{3a_2T_F}{2} + \frac{20a_sC_AT_F}{3} + 4a_sC_FT_F\right)\right. \\
&\left.\left.- \frac{17a_sC_A^2}{3} - 2T_F\mathcal{Y}_d - 2T_F\mathcal{Y}_u\right]\right\} \\
&+ a_s\left\{\frac{1}{\epsilon^3}\left[-\frac{1331}{27}a_s^2C_A^3 + \frac{968}{9}a_s^2C_A^2T_Fn_G - \frac{704}{9}a_s^2C_AT_F^2n_G^2 + \frac{512}{27}a_s^2T_F^3n_G^3\right]\right. \\
&+ \frac{1}{\epsilon^2}\left[n_G\left(\frac{11a_1^2T_F}{900} - \frac{121}{45}a_1a_sC_AT_F - \frac{43a_2^2T_F}{12} - 11a_2a_sC_AT_F - \frac{2492}{27}a_s^2C_A^2T_F\right.\right. \\
&\left.- \frac{308}{9}a_s^2C_AC_FT_F - \frac{32}{3}a_sT_F^2\mathcal{Y}_d - \frac{32}{3}a_sT_F^2\mathcal{Y}_u\right) + n_G^2\left(\frac{121a_1^2T_FN_c}{4050} + \frac{11a_1^2T_F}{150}\right. \\
&\left.+ \frac{88}{45}a_1a_sT_F^2 + \frac{1}{6}a_2^2T_FN_c + \frac{a_2^2T_F}{6} + 8a_2a_sT_F^2 + \frac{1120}{27}a_s^2C_AT_F^2 + \frac{224}{9}a_s^2C_FT_F^2\right) \\
&+ \frac{a_1T_F\mathcal{Y}_d}{6} + \frac{17a_1T_F\mathcal{Y}_u}{30} + \frac{3a_2T_F\mathcal{Y}_d}{2} + \frac{3a_2T_F\mathcal{Y}_u}{2} + \frac{1309a_s^2C_A^3}{27} + \frac{44}{3}a_sC_AT_F\mathcal{Y}_d \\
&+ \frac{44}{3}a_sC_AT_F\mathcal{Y}_u + 4a_sC_FT_F\mathcal{Y}_d + 4a_sC_FT_F\mathcal{Y}_u - \frac{2}{3}T_FN_c\mathcal{Y}_d^2 - \frac{4}{3}T_FN_c\mathcal{Y}_d\mathcal{Y}_u \\
&\left.- \frac{2}{3}T_FN_c\mathcal{Y}_u^2 - \frac{2T_F\mathcal{Y}_d\mathcal{Y}_l}{3} - T_F\mathcal{Y}_{dd} - \frac{2T_F\mathcal{Y}_l\mathcal{Y}_u}{3} + 2T_F\mathcal{Y}_{ud} - T_F\mathcal{Y}_{uu}\right] \\
&+ \frac{1}{\epsilon}\left[n_G\left(-\frac{13a_1^2T_F}{180} - \frac{a_1a_2T_F}{60} + \frac{22}{45}a_1a_sC_AT_F - \frac{11}{45}a_1a_sC_FT_F + \frac{241a_2^2T_F}{36}\right.\right. \\
&\left.+ 2a_2a_sC_AT_F - a_2a_sC_FT_F + \frac{2830}{81}a_s^2C_A^2T_F + \frac{410}{27}a_s^2C_AC_FT_F - \frac{4}{3}a_s^2C_F^2T_F\right) \\
&+ n_G^2\left(-\frac{1331a_1^2T_FN_c}{24300} - \frac{121a_1^2T_F}{900} - \frac{11}{36}a_2^2T_FN_c - \frac{11a_2^2T_F}{36} - \frac{632}{81}a_s^2C_AT_F^2\right. \\
&\left.- \frac{176}{27}a_s^2C_FT_F^2\right) - \frac{89a_1T_F\mathcal{Y}_d}{60} - \frac{101a_1T_F\mathcal{Y}_u}{60} - \frac{31a_2T_F\mathcal{Y}_d}{4} - \frac{31a_2T_F\mathcal{Y}_u}{4} \\
&- \frac{2857}{162}a_s^2C_A^3 - 8a_sC_AT_F\mathcal{Y}_d - 8a_sC_AT_F\mathcal{Y}_u - 2a_sC_FT_F\mathcal{Y}_d - 2a_sC_FT_F\mathcal{Y}_u
\end{aligned}
$$

$$
+\frac{7}{3}T_F N_c \mathcal{Y}_d^2 + \frac{14}{3}T_F N_c \mathcal{Y}_d \mathcal{Y}_u + \frac{7}{3}T_F N_c \mathcal{Y}_u^2 + \frac{7T_F\mathcal{Y}_d\mathcal{Y}_l}{3} + 3T_F\mathcal{Y}_{dd} + \frac{7T_F\mathcal{Y}_l\mathcal{Y}_u}{3}
$$

$$
-2T_F\mathcal{Y}_{ud} + 3T_F\mathcal{Y}_{uu}\Bigg]\Bigg\}, \tag{41}
$$

$$
\begin{aligned}
Z_{\xi_B} &= 1 + a_1\frac{1}{\epsilon}\Bigg\{n_G\Bigg(-\frac{11N_c}{45}-\frac{3}{5}\Bigg)-\frac{1}{10}\Bigg\}\\
&+ a_1\Bigg\{\frac{1}{\epsilon}\Bigg[n_G\Bigg(-\frac{137a_1N_c}{1800}-\frac{81a_1}{200}-\frac{a_2N_c}{40}-\frac{9a_2}{40}-\frac{11a_sC_FN_c}{30}\Bigg)\\
&-\frac{9a_1}{100}-\frac{9a_2}{20}+\frac{N_c\mathcal{Y}_d}{12}+\frac{17N_c\mathcal{Y}_u}{60}+\frac{3\mathcal{Y}_l}{4}\Bigg]\Bigg\}\\
&+ a_1\Bigg\{\frac{1}{\epsilon^2}\Bigg[n_G\Bigg(-\frac{533a_1^2N_c}{54000}-\frac{63a_1^2}{2000}+\frac{7a_2^2N_c}{720}+\frac{39a_2^2}{80}+\frac{121}{270}a_s^2C_AC_FN_c\Bigg)\\
&-n_G^2\Bigg(\frac{1507a_1^2N_c^2}{243000}+\frac{217a_1^2N_c}{4500}+\frac{81a_1^2}{1000}+\frac{a_2^2N_c^2}{360}+\frac{a_2^2N_c}{36}+\frac{a_2^2}{40}+\frac{44}{135}a_s^2C_FT_FN_c\Bigg)\\
&-\frac{3a_1^2}{1000}-\frac{a_1N_c\mathcal{Y}_d}{144}-\frac{289a_1N_c\mathcal{Y}_u}{3600}-\frac{9a_1\mathcal{Y}_l}{16}+\frac{43a_2^2}{40}-\frac{a_2N_c\mathcal{Y}_d}{16}-\frac{17a_2N_c\mathcal{Y}_u}{80}\\
&-\frac{9a_2\mathcal{Y}_l}{16}-\frac{1}{6}a_sC_FN_c\mathcal{Y}_d-\frac{17}{30}a_sC_FN_c\mathcal{Y}_u+\frac{N_c^2\mathcal{Y}_d^2}{36}+\frac{11}{90}N_c^2\mathcal{Y}_d\mathcal{Y}_u+\frac{17N_c^2\mathcal{Y}_u^2}{180}\\
&+\frac{5N_c\mathcal{Y}_d\mathcal{Y}_l}{18}+\frac{N_c\mathcal{Y}_{dd}}{24}+\frac{31N_c\mathcal{Y}_l\mathcal{Y}_u}{90}-\frac{11N_c\mathcal{Y}_{ud}}{60}+\frac{17N_c\mathcal{Y}_{uu}}{120}+\frac{\mathcal{Y}_l^2}{4}+\frac{3\mathcal{Y}_{ll}}{8}\Bigg]\\
&+\frac{1}{\epsilon}\Bigg[n_G\Bigg(\frac{1697a_1^2N_c}{54000}+\frac{327a_1^2}{2000}+\frac{a_1a_2N_c}{3600}+\frac{9a_1a_2}{400}+\frac{137a_1a_sC_FN_c}{2700}-\frac{a_2^2N_c}{135}\\
&-\frac{9a_2^2}{10}+\frac{1}{60}a_2a_sC_FN_c-\frac{1463a_s^2C_AC_FN_c}{1620}+\frac{11}{90}a_s^2C_F^2N_c\Bigg)+n_G^2\Bigg(\frac{16577a_1^2N_c^2}{1458000}\\
&+\frac{2387a_1^2N_c}{27000}+\frac{297a_1^2}{2000}+\frac{11a_2^2N_c^2}{2160}+\frac{11a_2^2N_c}{216}+\frac{11a_2^2}{240}+\frac{242}{405}a_s^2C_FT_FN_c\Bigg)\\
&-\frac{163a_1^2}{8000}-\frac{261a_1a_2}{800}-\frac{18a_1\lambda}{25}+\frac{1267a_1N_c\mathcal{Y}_d}{7200}+\frac{2827a_1N_c\mathcal{Y}_u}{7200}+\frac{843a_1\mathcal{Y}_l}{800}\\
&-\frac{3401a_2^2}{960}-\frac{6a_2\lambda}{5}+\frac{437a_2N_c\mathcal{Y}_d}{480}+\frac{157a_2N_c\mathcal{Y}_u}{96}+\frac{543a_2\mathcal{Y}_l}{160}+\frac{17}{60}a_sC_FN_c\mathcal{Y}_d\\
&+\frac{29}{60}a_sC_FN_c\mathcal{Y}_u+\frac{48\lambda^2}{5}-\frac{17N_c^2\mathcal{Y}_d^2}{360}-\frac{59}{180}N_c^2\mathcal{Y}_d\mathcal{Y}_u-\frac{101N_c^2\mathcal{Y}_u^2}{360}-\frac{33\mathcal{Y}_l^2}{40}\\
&-\frac{157N_c\mathcal{Y}_d\mathcal{Y}_l}{180}-\frac{61N_c\mathcal{Y}_{dd}}{240}-\frac{199N_c\mathcal{Y}_l\mathcal{Y}_u}{180}-\frac{N_c\mathcal{Y}_{ud}}{24}
\end{aligned}
$$

$$
-\frac{113N_c\mathcal{Y}_{uu}}{240}-\frac{87\mathcal{Y}_{ll}}{80}\Bigg]\Bigg\}, \tag{42}
$$

$$
\begin{aligned}
Z_{\xi_W} = {} & 1 + a_2\frac{1}{\epsilon}\Bigg\{-\xi_W + n_G\bigg(-\frac{N_c}{3}-\frac{1}{3}\bigg)+\frac{25}{6}\Bigg\} \\
& + a_2\Bigg\{\frac{1}{\epsilon^2}\Bigg[a_2\xi_W^2 + n_G\bigg(\frac{a_2\xi_W N_c}{3}+\frac{a_2\xi_W}{3}+\frac{a_2N_c}{2}+\frac{a_2}{2}\bigg)-\frac{8a_2\xi_W}{3}-\frac{25a_2}{4}\Bigg] \\
& + \frac{1}{\epsilon}\Bigg[n_G\bigg(-\frac{a_1N_c}{120}-\frac{3a_1}{40}-\frac{13a_2N_c}{8}-\frac{13a_2}{8}-\frac{a_sC_FN_c}{2}\bigg) \\
& - \frac{3a_1}{20}-\frac{a_2\xi_W^2}{2}-\frac{11a_2\xi_W}{4}+\frac{113a_2}{8}+\frac{N_c\mathcal{Y}_d}{4}+\frac{N_c\mathcal{Y}_u}{4}+\frac{\mathcal{Y}_l}{4}\Bigg]\Bigg\} \\
& + a_2\Bigg\{\frac{1}{\epsilon^3}\Bigg[-a_2^2\xi_W^3 + n_G\bigg(-\frac{1}{3}a_2^2\xi_W^2N_c-\frac{1}{3}a_2^2\xi_W^2-\frac{5}{6}a_2^2\xi_WN_c-\frac{5a_2^2\xi_W}{6} \\
& - \frac{43a_2^2N_c}{18}-\frac{43a_2^2}{18}\bigg)+\frac{7a_2^2\xi_W^2}{6}+\frac{89a_2^2\xi_W}{12}+n_G^2\bigg(\frac{a_2^2N_c^2}{18}+\frac{a_2^2N_c}{9}+\frac{a_2^2}{18}\bigg) \\
& + \frac{1525a_2^2}{72}\Bigg]+\frac{1}{\epsilon^2}\Bigg[n_G\bigg(-\frac{a_1^2N_c}{80}-\frac{13a_1^2}{400}+\frac{1}{120}a_1a_2\xi_WN_c+\frac{3a_1a_2\xi_W}{40} \\
& + \frac{a_1a_2N_c}{120}+\frac{3a_1a_2}{40}+\frac{1}{6}a_2^2\xi_W^2N_c+\frac{a_2^2\xi_W^2}{6}+\frac{47}{24}a_2^2\xi_WN_c+\frac{47a_2^2\xi_W}{24} \\
& + \frac{4273a_2^2N_c}{432}+\frac{4273a_2^2}{432}+\frac{1}{2}a_2a_sC_F\xi_WN_c+\frac{1}{2}a_2a_sC_FN_c+\frac{11}{18}a_s^2C_AC_FN_c\bigg) \\
& + n_G^2\bigg(-\frac{11a_1^2N_c^2}{16200}-\frac{7a_1^2N_c}{900}-\frac{3a_1^2}{200}-\frac{59a_2^2N_c^2}{216}-\frac{59a_2^2N_c}{108}-\frac{59a_2^2}{216} \\
& - \frac{4}{9}a_s^2C_FT_FN_c\bigg)-\frac{a_1^2}{200}+\frac{3a_1a_2\xi_W}{20}+\frac{3a_1a_2}{20}-\frac{a_1N_c\mathcal{Y}_d}{48}-\frac{17a_1N_c\mathcal{Y}_u}{240} \\
& - \frac{3a_1\mathcal{Y}_l}{16}+\frac{7a_2^2\xi_W^3}{6}+\frac{53a_2^2\xi_W^2}{12}-\frac{271a_2^2\xi_W}{24}-\frac{29629a_2^2}{432}-\frac{1}{4}a_2\xi_WN_c\mathcal{Y}_d \\
& - \frac{1}{4}a_2\xi_WN_c\mathcal{Y}_u-\frac{a_2\xi_W\mathcal{Y}_l}{4}-\frac{7a_2N_c\mathcal{Y}_d}{16}-\frac{7a_2N_c\mathcal{Y}_u}{16}-\frac{7a_2\mathcal{Y}_l}{16}-\frac{1}{2}a_sC_FN_c\mathcal{Y}_d \\
& - \frac{1}{2}a_sC_FN_c\mathcal{Y}_u+\frac{N_c^2\mathcal{Y}_d^2}{12}+\frac{1}{6}N_c^2\mathcal{Y}_d\mathcal{Y}_u+\frac{N_c^2\mathcal{Y}_u^2}{12}+\frac{N_c\mathcal{Y}_d\mathcal{Y}_l}{6}+\frac{N_c\mathcal{Y}_{dd}}{8} \\
& + \frac{N_c\mathcal{Y}_l\mathcal{Y}_u}{6}-\frac{N_c\mathcal{Y}_{ud}}{4}+\frac{N_c\mathcal{Y}_{uu}}{8}+\frac{\mathcal{Y}_l^2}{12}+\frac{\mathcal{Y}_{ll}}{8}\Bigg]+\frac{1}{\epsilon}\Bigg[n_G\bigg(\frac{287a_1^2N_c}{10800}+\frac{91a_1^2}{1200} \\
& - \frac{1}{15}a_1a_2N_c\zeta(3)+\frac{2a_1a_2N_c}{45}-\frac{3a_1a_2\zeta(3)}{5}+\frac{2a_1a_2}{5}+\frac{1}{180}a_1a_sC_FN_c
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2}{3}a_2^2\xi_W N_c+\frac{2a_2^2\xi_W}{3}+3a_2^2N_c\zeta(3)-\frac{7025a_2^2N_c}{432}+3a_2^2\zeta(3)-\frac{7025a_2^2}{432}\\
&-4a_2a_sC_FN_c\zeta(3)+\frac{8}{3}a_2a_sC_FN_c-\frac{133}{108}a_s^2C_AC_FN_c+\frac{1}{6}a_s^2C_F^2N_c\Big)\\
&+n_G^2\bigg(\frac{121a_1^2N_c^2}{97200}+\frac{77a_1^2N_c}{5400}+\frac{11a_1^2}{400}+\frac{185a_2^2N_c^2}{432}+\frac{185a_2^2N_c}{216}+\frac{185a_2^2}{432}\\
&+\frac{22}{27}a_s^2C_FT_FN_c\bigg)-\frac{163a_1^2}{4800}-\frac{3a_1a_2\zeta(3)}{10}-\frac{11a_1a_2}{32}-\frac{2a_1\lambda}{5}+\frac{533a_1N_c\mathcal{Y}_d}{1440}\\
&+\frac{593a_1N_c\mathcal{Y}_u}{1440}+\frac{17a_1\mathcal{Y}_l}{32}-\frac{7}{12}a_2^2\xi_W^3-\frac{1}{2}a_2^2\xi_W^2\zeta(3)-\frac{11a_2^2\xi_W^2}{4}-2a_2^2\xi_W\zeta(3)\\
&-\frac{105a_2^2\xi_W}{8}+\frac{a_2^2\zeta(3)}{2}+\frac{143537a_2^2}{1728}-2a_2\lambda+\frac{79a_2N_c\mathcal{Y}_d}{96}+\frac{79a_2N_c\mathcal{Y}_u}{96}\\
&+\frac{79a_2\mathcal{Y}_l}{96}+\frac{7}{12}a_sC_FN_c\mathcal{Y}_d+\frac{7}{12}a_sC_FN_c\mathcal{Y}_u+16\lambda^2-\frac{5N_c^2\mathcal{Y}_d^2}{24}-\frac{5}{12}N_c^2\mathcal{Y}_d\mathcal{Y}_u\\
&-\frac{5N_c^2\mathcal{Y}_u^2}{24}-\frac{5N_c\mathcal{Y}_d\mathcal{Y}_l}{12}-\frac{19N_c\mathcal{Y}_{dd}}{48}-\frac{5N_c\mathcal{Y}_l\mathcal{Y}_u}{12}-\frac{3N_c\mathcal{Y}_{ud}}{8}-\frac{19N_c\mathcal{Y}_{uu}}{48}\\
&-\frac{5\mathcal{Y}_l^2}{24}-\frac{19\mathcal{Y}_{ll}}{48}\bigg]\bigg\}\,, \qquad (43)
\end{aligned}
$$

$$
\begin{aligned}
Z_{\xi_G}&=1+a_s\frac{1}{\epsilon}\bigg\{-\frac{C_A\xi_G}{2}+\frac{13C_A}{6}-\frac{8T_Fn_G}{3}\bigg\}\\
&+a_s\bigg\{\frac{1}{\epsilon^2}\bigg[\frac{1}{4}a_sC_A^2\xi_G^2-\frac{17}{24}a_sC_A^2\xi_G-\frac{13a_sC_A^2}{8}\\
&+n_G\left(\frac{4}{3}a_sC_A\xi_GT_F+2a_sC_AT_F\right)\bigg]\\
&+\frac{1}{\epsilon}\bigg[n_G\bigg(-\frac{11a_1T_F}{30}-\frac{3a_2T_F}{2}-5a_sC_AT_F-4a_sC_FT_F\bigg)\\
&-\frac{1}{8}a_sC_A^2\xi_G^2-\frac{11}{16}a_sC_A^2\xi_G+\frac{59a_sC_A^2}{16}+2T_F\mathcal{Y}_d+2T_F\mathcal{Y}_u\bigg]\bigg\}\\
&+a_s\bigg\{\frac{1}{\epsilon^3}\bigg[-\frac{1}{8}a_s^2C_A^3\xi_G^3+\frac{1}{6}a_s^2C_A^3\xi_G^2+\frac{47}{48}a_s^2C_A^3\xi_G+\frac{403a_s^2C_A^3}{144}\\
&+n_G\bigg(-\frac{2}{3}a_s^2C_A^2\xi_G^2T_F-\frac{5}{3}a_s^2C_A^2\xi_GT_F-\frac{44}{9}a_s^2C_A^2T_F\bigg)+\frac{16}{9}a_s^2C_AT_F^2n_G^2\bigg]\\
&+\frac{1}{\epsilon^2}\bigg[n_G\bigg(-\frac{11a_1^2T_F}{900}+\frac{11}{60}a_1a_sC_A\xi_GT_F+\frac{11}{60}a_1a_sC_AT_F+\frac{43a_2^2T_F}{12}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{3}{4}a_2 a_s C_A \xi_G T_F + \frac{3}{4}a_2 a_s C_A T_F + \frac{1}{3}a_s^2 C_A^2 \xi_G^2 T_F + \frac{19}{6}a_s^2 C_A^2 \xi_G T_F \\
&+ \frac{481}{27}a_s^2 C_A^2 T_F + 2a_s^2 C_A C_F \xi_G T_F + \frac{62}{9}a_s^2 C_A C_F T_F\bigg) + n_G^2\bigg(- \frac{121a_1^2 T_F N_c}{4050} \\
&- \frac{11a_1^2 T_F}{150} - \frac{1}{6}a_2^2 T_F N_c - \frac{a_2^2 T_F}{6} - \frac{200}{27}a_s^2 C_A T_F^2 - \frac{32}{9}a_s^2 C_F T_F^2\bigg) - \frac{a_1 T_F \mathcal{Y}_d}{6} \\
&- \frac{17a_1 T_F \mathcal{Y}_u}{30} - \frac{3a_2 T_F \mathcal{Y}_d}{2} - \frac{3a_2 T_F \mathcal{Y}_u}{2} + \frac{7}{48}a_s^2 C_A^3 \xi_G^3 + \frac{13}{24}a_s^2 C_A^3 \xi_G^2 \\
&- \frac{143}{96}a_s^2 C_A^3 \xi_G - \frac{7957a_s^2 C_A^3}{864} - a_s C_A \xi_G T_F \mathcal{Y}_d - a_s C_A \xi_G T_F \mathcal{Y}_u - a_s C_A T_F \mathcal{Y}_d \\
&- a_s C_A T_F \mathcal{Y}_u - 4a_s C_F T_F \mathcal{Y}_d - 4a_s C_F T_F \mathcal{Y}_u + \frac{2}{3}T_F N_c \mathcal{Y}_d^2 + \frac{4}{3}T_F N_c \mathcal{Y}_d \mathcal{Y}_u \\
&+ \frac{2}{3}T_F N_c \mathcal{Y}_u^2 + \frac{2T_F \mathcal{Y}_d \mathcal{Y}_l}{3} + T_F \mathcal{Y}_{dd} + \frac{2T_F \mathcal{Y}_l \mathcal{Y}_u}{3} - 2T_F \mathcal{Y}_{ud} + T_F \mathcal{Y}_{uu}\bigg] \\
&+ \frac{1}{\epsilon}\bigg[n_G\bigg(\frac{13a_1^2 T_F}{180} + \frac{a_1 a_2 T_F}{60} - \frac{22}{15}a_1 a_s C_A T_F \zeta(3) + \frac{319}{360}a_1 a_s C_A T_F \\
&+ \frac{11}{45}a_1 a_s C_F T_F - \frac{241a_2^2 T_F}{36} - 6a_2 a_s C_A T_F \zeta(3) + \frac{29}{8}a_2 a_s C_A T_F + a_2 a_s C_F T_F \\
&+ \frac{4}{3}a_s^2 C_A^2 \xi_G T_F + 12a_s^2 C_A^2 T_F \zeta(3) - \frac{911}{27}a_s^2 C_A^2 T_F - 16a_s^2 C_A C_F T_F \zeta(3) \\
&- \frac{5}{27}a_s^2 C_A C_F T_F + \frac{4}{3}a_s^2 C_F^2 T_F\bigg) + n_G^2\bigg(\frac{1331a_1^2 T_F N_c}{24300} + \frac{121a_1^2 T_F}{900} + \frac{11}{36}a_2^2 T_F N_c \\
&+ \frac{11a_2^2 T_F}{36} + \frac{304}{27}a_s^2 C_A T_F^2 + \frac{176}{27} + a_s^2 C_F T_F^2\bigg) + \frac{89a_1 T_F \mathcal{Y}_d}{60} + \frac{101a_1 T_F \mathcal{Y}_u}{60} \\
&+ \frac{31a_2 T_F \mathcal{Y}_d}{4} + \frac{31a_2 T_F \mathcal{Y}_u}{4} - \frac{7}{96}a_s^2 C_A^3 \xi_G^3 - \frac{1}{16}a_s^2 C_A^3 \xi_G^2 \zeta(3) - \frac{11}{32}a_s^2 C_A^3 \xi_G^2 \\
&- \frac{1}{4}a_s^2 C_A^3 \xi_G \zeta(3) - \frac{167}{96}a_s^2 C_A^3 \xi_G - \frac{3}{16}a_s^2 C_A^3 \zeta(3) + \frac{9965a_s^2 C_A^3}{864} + \frac{25}{6}a_s C_A T_F \mathcal{Y}_d \\
&+ \frac{25}{6}a_s C_A T_F \mathcal{Y}_u + 2a_s C_F T_F \mathcal{Y}_d + 2a_s C_F T_F \mathcal{Y}_u - \frac{7}{3}T_F N_c \mathcal{Y}_d^2 - \frac{14}{3}T_F N_c \mathcal{Y}_d \mathcal{Y}_u \\
&- \frac{7}{3}T_F N_c \mathcal{Y}_u^2 - \frac{7T_F \mathcal{Y}_d \mathcal{Y}_l}{3} - 3T_F \mathcal{Y}_{dd} - \frac{7T_F \mathcal{Y}_l \mathcal{Y}_u}{3} \\
&+ 2T_F \mathcal{Y}_{ud} - 3T_F \mathcal{Y}_{uu}\bigg]\bigg\}\,. \qquad (44)
\end{aligned}
$$

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